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# On random fixed point theorems with applications to integral equations

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ABSTRACT

This particular research establishes some random fixed point theorems for general nonlinear random contractive operators in the context of partially ordered separable metric spaces. The existence and uniqueness of the random solution for the nonlinear integral equation is obtained by applying the result of the random fixed point. The results generalize and improve on some related works in the literature.

- Our theorems are proved in the context of metric space while Saluja and Tripathi [8] proved in the context of partial metric spaces.
- Nieto, Ouahab and. Rodriguez-Lopez [9] proved their theorem using Banach contraction mappings while we proved our theorems using Hardy and Rogers contraction mappings.
- Rashwan and Albaqeri [3] proved the solution of the random integral equation using the Banach contraction operator while we proved our solution to the random integral equations employing more general contractive operator.

## 1. Introduction

Fixed point theory plays very fundamental role in solving deterministic operator equations. An important principle that is required to solve applied mathematical problems in applied mathematics, such as computer sciences involve looping. Fixed point iteration and monotonous iteration techniques introduced by Banach [1] and Ran and Reurings [2] respectively handled these problems.

Probabilistic fundamental analysis is an important aspect of mathematics that is applied to solving problems. An equation that requires a mathematical model to describe its phenomena is classified as random equation. Fixed point theorems for stochastic functions was pioneered by Prague School of Probability. In 1955–1965, Spacek and Hans studied the Fredholm integral equations with respect to random kernel. Stochastic fixed point theorem for contractive mappings in Polish spaces were established by Rashwan and Albaqeri [3], Hans [4], Hans and Spacek [5], Okeke and Eke [6]. In 1977, Lee and Padgett [7] proved fixed point result for stochastic nonlinear contractive mappings in separable Banach spaces and applied the result obtained to establish the existence and uniqueness of the random solution of the stochastic nonlinear integral equations. Saluja and Tripathi [8] proved some stochastic fixed point results for contractive mappings in the framework of cone random metric spaces. Nieto et al. [9] established a stochastic fixed point theorem for Banach contraction mappings in ordered metric spaces. In the same reference, the result obtained is used to established the solution for random differential equations with boundary properties.

This particular paper proves the existence and uniqueness of random fixed point for a more general nonlinear contractive functions in partially ordered separable metric spaces. The result obtain is apply to establish the existence and uniqueness of random solution for nonlinear integral equation.

## 2. Methodology

Suppose $(A, \xi)$ is a separable Banach space, where $\xi$ is a$\sigma$-algebra of Borel subsets of A and $(\phi, \xi, \mu)$ represents a complete probability measure space with measure$\mu$. $\xi$ is also a $\sigma$-algebra of subsets of $\phi$. The following definitions are found in Joshi and Bose [10] with more details for the readers.

**Definition 1.1.** An operator $T : \phi \times A \to B$ is known as a random operator if $T(v, a) = B(v)$ is a random variable for every $a \in A$.

**Definition 1.2.** An operator $T : \phi \times A \to B$ is called continuous random mapping if the set of all $v \in \phi$ for which $T(v, a)$ is a continuous function of

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a has measure one.

**Definition 1.3**. An equation of the type $T(v, a(v)) = a(v)$, where $T : \phi \times A \to A$ is a random mapping is called a random fixed point equation.

**Definition 1.4**. Any mapping $a : \phi \to A$ which satisfies random fixed point equation $T(v, a(v)) = a(v)$ almost surely is called a wide sense solution of the fixed point equation.

**Definition 1.5**. Any A-valued random variable $a(v)$ which satisfies $\mu\{v : T(v, a(v)) = a(v)\} = 1$ is called random solution of the fixed point equation.

Let A be a nonempty set and let $T : A \to A$ be an operator. We define the Picard iteration of T by $T^n = T(T^{n-1})$, for $n \in \mathbb{N}, n \geq 2$.

**Definition 1.6**. [9]: If $(A, \leq)$is a partially ordered set and $T : A \to A$ , we say that T is monotone non decreasing if $a \leq b, a, b \in A \Rightarrow T(a) \leq T(b)$.

## 3. Results

This section proves the existence and uniqueness of random fixed point for contractive mappings in partially ordered separable metric spaces.

**Theorem 3.1**. Let $(\phi, T)$ be a measurable space and $(A, d, \alpha)$ a complete partially ordered separable metric space. If $T : \phi \times A \to A$ is a continuous random operator such that, $T(v, \cdot)$ is a monotonous operator. If the following conditions hold:

($J_1$) For each $v \in \phi$, there exists $m(v) + n(v) + 2q(v) < 1$ such that

$$\begin{aligned} &d(T(v; a_1), T(v; a_2)) \leq m(v)d(a_1, a_2) \\ &+n(v)[d(a_1, T(v; a_1)) + d(a_2, T(v; a_2))] \\ &+q(v)[d(a_1, T(v; a_2)) + d(a_2, T(v; a_1))]. \end{aligned} \tag{1}$$

For each $a_1, a_2 \in A,\ a_1 \geq a_2$.

($J_2$) There is a random variable $a_0 : \phi \to A$ with

$$a_0(v) \leq T(v, a_0(v)),\ \text{for all v} \in \phi,$$

or

$$a_0(v) \geq T(v, a_0(v)),\ \text{for all v} \in \phi.$$

Then there is a random variable $a : \phi \to A$ which is a random fixed point of T.

**Proof:** If for $v \in \phi$, there is a random variable $a_0(v) \in (\phi, T)$such that $T(v, a_0(v)) = a_0(v)$then $a_0(v)$is a random fixed point of T. On the contrary, we assume that $T(v; a_0(v)) \neq a_0(v)$ for some $v \in \phi$. Let $b_0(v) = a_0(v)$, then we can define a sequence $T(v, b_{n-1}(v)) = b_n(v)$, for $v \in \phi, n \in \mathbb{N}$. According to ($J_1$) we have, for each $v \in \phi$, and $n \in \mathbb{N}$ one of the following relations hold:

$b_n(v) \leq b_{n+1}(v)$ or $b_n(v) \geq b_{n+1}(v)$.

Using the above inequality in Eq. (1) we obtain,

$$\begin{aligned} &d(b_n(v), b_{n+1}(v)) \leq d(T(v; b_{n-1}(v)), T(v; b_n(v))) \\ &\leq m(v)d(b_{n-1}(v), b_n(v)) \\ &+n(v)[d(b_{n-1}(v), T(v; b_{n-1}(v))) + d(b_n(v), T(v; b_n(v)))] \\ &+q(v)[d(b_{n-1}(v), T(v; b_n(v))) + d(b_n(v), T(v; b_{n-1}(v)))] \end{aligned}$$

$$\begin{aligned} &\leq m(v)d(b_{n-1}(v), b_n(v)) \\ &+n(v)[d(b_{n-1}(v), b_n(v)) + d(b_n(v), b_{n+1}(v))] \\ &+q(v)[d(b_{n-1}(v), b_{n+1}(v)) + d(b_n(v), b_n(v))] \end{aligned}$$

$$\begin{aligned} &\leq m(v)d(b_{n-1}(v), b_n(v)) \\ &+n(v)[d(b_{n-1}(v), b_n(v)) + d(b_n(v), b_{n+1}(v))] \\ &+q(v)[d(b_{n-1}(v), b_n(v)) + d(b_n(v), b_{n+1}(v))] \end{aligned}$$

$$\begin{aligned} &\leq (m(v) + n(v) + q(v))d(b_{n-1}(v), b_n(v)) \\ &+ (n(v) + q(v))d(b_n(v), b_{n+1}(v))] \end{aligned}$$

$$\leq \frac{m(v) + n(v) + q(v)}{1 - n(v) - q(v)} d(b_{n-1}(v), b_n(v))$$

$$\leq kd(b_{n-1}(v), b_n(v)). \tag{2}$$

Where $k = \frac{m(v)+n(v)+q(v)}{1-n(v)-q(v)} < 1$.

Consequently, we get

$$d(b_n(v), b_{n+1}(v)) \leq [k(v)]^n d(b_0(v), b_1(v)).$$

For n > m we obtain,

$$\begin{aligned} &d(b_m(v), b_n(v)) \leq d(b_n(v), b_{m+1}(v)) + d(b_{m+1}(v), b_{m+2}(v)) \\ &+ \cdots + d(b_{n-1}(v), b_n(v)) \\ &\leq ([k(v)]^m + [k(v)]^{m+1} + \cdots + [k(v)]^{m+n-1})d(b_0(v), b_1(v)) \\ &\leq \frac{[k(v)]^m}{1 - k(v)} d(b_0(v), b_1(v)) \end{aligned}$$

This shows that $\{b_n(v)\}_{n \in \mathbb{N}}$ is a Cauchy sequence for every $v \in \phi$. Let $b(v)$ be the limit point of $\{b_n(v)\}$, $v \in \phi$. Since $b_0(v)$ is measurable, then $b_1(v)$ is measurable. Consequently, for each $n \in \mathbb{N}$, the operator $v \to b_n(v)$ is measurable. This implies that $b(v)$ is measurable.

Next we prove that $b(v)$ is the random fixed point of T.

$$\begin{aligned} &d(b(v), T(v; b(v)) \leq d(b(v), b_n(v)) + d(b_n(v), T(v; b(v)) \\ &\leq d(b(v), b_n(v)) + d(T(v; b_{n-1}(v)), T(v; b(v))) \end{aligned}$$

$$\begin{aligned} &\leq d(b(v), b_n(v)) + m(v)d(b_{n-1}(v), b(v)) \\ &+ n(v)[d(b_{n-1}(v), T(v; b_{n-1}(v)) + d(b(v), T(v; b(v))] \\ &+ q(v)[d(b_{n-1}(v), T(v; b(v)) + d(b(v), T(v; b_{n-1}(v))]. \end{aligned}$$

As $n \to \infty$, we get

$$\begin{aligned} &d(b(v), T(v; b(v)) \leq d(b(v), (v)) + m(v)d(b(v), b(v)) \\ &+ n(v)[d(b(v), T(v; b(v)) + d(b(v), T(v; b(v))] \\ &+ q(v)[d(b(v), T(v; b(v)) + d(b(v), T(v; b(v))] \end{aligned}$$

$$\leq (2n(v) + 2q(v))d(b(v), T(v; b(v))$$

Since $2n(v) + 2q(v) < 1$ then it is a contradiction. Therefore $b(v)$ is the random fixed point of T.

To prove the uniqueness of the random fixed point of T, we consider the following proposition in [2].

($J_3$) Every pair of elements of A has a lower bound or upper bound.

In [2], the above condition [$J_3$] is equivalent to:

[$J_3$'] for every $a, b \in A$, there is $c \in A$ that is comparable to a and b.

The following theorem concludes that the existence of a random variable $b : \phi \to A$ in Theorem 3.1 is the unique random fixed point of T.

**Theorem 3.2**. Let all the hypotheses of Theorem 3.1 be satisfied and if the following condition [$J_3$] (equivalently [$J_3$']) be satisfied, then $b : \phi \to A$ is the unique random fixed point of T.

**Proof:** Let $b'$ : $\phi \to A$ be an arbitrary random variable and we define the sequence

$$b'_0(v) = a'_0(v),\ b'_n(v) = T\left(v; b'_{n-1}(v)\right),\ \ v \in \phi,\ n \in \mathbb{N}.$$

In Theorem 3.1, we obtain that $\{b_n(v)\}_{n \in \mathbb{N}} \to b(v)$ as $n \to \infty$, for each $v \in \phi$, where $b(v)$ is the random fixed point of T. If $b'_0(v)$ is comparable to $b_0(v)$ for each $v \in \phi$, it follows that $T(v; b_0(v))'$ is comparable to $T(v; b_0(v))$ for each $v \in \phi$. Therefore $b'_n(v)$ is comparable to $b_n(v)$ for each $v \in \phi$.

Hence,

$$d\left(b_n(v),b'_n(v)\right)\le d\left(T(v;b_{n-1}(v)),T\left(v;b'_{n-1}(v)\right)\right)$$
$$\le m(v)d\left(b_{n-1}(v),b'_{n-1}(v)\right)$$
$$+n(v)[d(b_{n-1}(v),T(v;b_{n-1}(v)))$$
$$+d\left(b'_{n-1}(v),T\left(v;b'_{n-1}(v)\right)\right)\Big]$$
$$+q(v)\left[d\left(b_{n-1}(v),T\left(v;b'_{n-1}(v)\right)\right)\right.$$
$$\left.+d\left(b'_{n-1}(v),T(v;b_{n-1}(v))\right)\right]$$

$$= m(v)d\left(b_{n-1}(v),b'_{n-1}(v)\right)$$
$$+n(v)\left[d(b_{n-1}(v),b_n(v))+d\left(b'_{n-1}(v),b'_n(v)\right)\right]$$
$$+q(v)\left[d\left(b_{n-1}(v),b'_n(v)\right)+d\left(b'_{n-1}(v),b_n(v)\right)\right].$$

As $n\to\infty$, we get

$$d\left(b(v),b(v)'\right)\le m(v)d(b(v),b'(v))$$
$$+n(v)\left[d(b(v),b(v))+d(b'(v),b'(v))\right]$$
$$+q(v)\left[d(b(v),b'(v))+d(b'(v),b(v))\right]$$
$$\le (m(v)+2q(v))d(b(v),b'(v))$$

Since $m(v)+2q(v)<1$ then we have $b(v) = b'(v)$, for each $v\in\phi$.

On the contrary, choose an arbitrary random variable $b'_0 : \phi\to A$, for each $v\in\phi$, there is $c(v)\in A$ which is comparable to $b_0(v)$ and $b'_0(v)$ simultaneously. If we define

$$c_0(v)=c(v),\ \ c_n(v) = T(v;c_{n-1}(v)),\ v\in\phi,\ \mathrm{n}\in\mathbb{N}\,.$$

Then $b_n(v)$ is comparable to $c_n(v)$, for each $v\in\phi$.
Hence,

$$d(b_n(v)c_n(v))\le d(T(v;b_{n-1}(v)),T(v;c_{n-1}(v)))$$
$$\le m(v)d(b_{n-1}(v),c_{n-1}(v))$$
$$+n(v)[d(b_{n-1}(v),T(v;b_{n-1}(v)))$$
$$+d(c_{n-1}(v),T(v;c_{n-1}(v)))]$$
$$+q(v)[d(b_{n-1}(v),T(v;c_{n-1}(v)))$$
$$+d(c_{n-1}(v),T(v;b_{n-1}(v)))]$$

$$= m(v)d(b_{n-1}(v),c_{n-1}(v))$$
$$+n(v)[d(b_{n-1}(v),b_n(v))+d(c_{n-1}(v),c_n(v))]$$
$$+q(v)[d(b_{n-1}(v),c_n(v))+d(c_{n-1}(v),b_n(v))]$$

$$\le m(v)d(b_{n-1}(v),c_{n-1}(v))$$
$$+n(v)[d(b_{n-1}(v),b_n(v))+d(c_{n-1}(v),b_n(v))$$
$$+d(b_n(v),c_n(v))]$$
$$+q(v)[d(b_{n-1}(v),c_n(v))+d(c_{n-1}(v),b_n(v))].$$

As $n\to\infty$, we have

$$d(b(v),c_n(v))\le m(v)d(b(v),c_{n-1}(v))$$
$$+n(v)[d(c_{n-1}(v),b(v))+d(b(v),c_n(v))]$$
$$+q(v)[d(b(v),c_n(v))+d(c_{n-1}(v),b(v))]$$

$$\le (m(v)+n(v)+q(v))d(b(v),c_{n-1}(v))$$
$$+(n(v)+q(v))d(b(v),c_n(v))$$

$$\le\frac{m(v)+n(v)+q(v)}{1-n(v)-q(v)}d(b(v),c_{n-1}(v)).$$

Similarly, we obtain

$$d\left(b(v),c'_n(v)\right)\le\frac{m(v)+n(v)+q(v)}{1-n(v)-q(v)}d\left(b(v),c'_{n-1}(v)\right).$$

Choose a natural number $N_1$ such that

$$d(b(v),c_n(v))\le\frac{\varepsilon(1-n(v)-q(v))}{2(m(v)+n(v)+q(v))}$$
and
$$d\left(b(v),c'_n(v)\right)\le\frac{\varepsilon(1-n(v)-q(v))}{2(m(v)+n(v)+q(v))}. \tag{3}$$

Therefore,

$$d\left(b(v),b'(v)\right)\le d(b(v),c_{n-1}(v))+d\left(c_{n-1}(v),b'(v)\right)$$
$$\le\frac{m(v)+n(v)+q(v)}{1-n(v)-q(v)}\times\frac{\varepsilon(1-n(v)-q(v))}{2(m(v)+n(v)+q(v))}$$
$$+\frac{m(v)+n(v)+q(v)}{1-n(v)-q(v)}\times\frac{\varepsilon(1-n(v)-q(v))}{2(m(v)+n(v)+q(v))}$$
$$=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since $\varepsilon$ is the smallest positive number we assume $\varepsilon = 0$. Therefore $d(b(v),b'(v)) = 0$. Thus $b(v) = b'(v)$. The uniqueness proved.

**Example 3.3.** Let $A=[0,\infty)$ with the usual ordering and $\phi = [0,1]$, also $\xi$ a sigma algebra of Lebesgue measurable subset of [0, 1]. We define a mapping $d : \phi\times A\to A$ by

$$d(a(v),b(v))=|a(v)-b(v)|.$$

Then $(A,d,\le)$ is a complete partially ordered metric spaces. Define random operator $T : \phi\times A\to A$ by $T(c,a) = \frac{1-v^2+x}{4}$. We define a sequence of mappings $b_n : \phi\to A$ as $b_n(v)=(1-v^2)^{1+\left(1/n^2\right)}$ for every $v\in\phi$ and $n\in\mathbb{N}$. Define a measurable mapping $b : \phi\to A$ as $b(v) = (1-v^2)$ for each $v\in\phi$. Thus $(1-v^2)$ is the random fixed point of the random operator.

**Remark 3.4.** Theorem 3.1 is an extension of the result of Saluja and Tripathi [8] (Corollary 1) in the setting of partially ordered metric spaces.

If $b(v)=c(v) = 0$ in Theorem 3.1, then we obtain the result of Nieto et al. [9] Theorem 4.1.

**Corollary 3.5.** Let $(\phi,T)$ be a measurable space and $(A,d,\alpha)$ be a complete partially ordered separable metric space. If $T : \phi\times A\to A$ is a continuous random operator such that, $T(v,\cdot)$ is a monotone (either order-preserving or order-reserving) operator. Suppose that the following conditions hold:

($J_1$) For each $v\in\phi$, there exists $0\le m(v)<1$ such that

$$d(T(v;a_1),T(v;a_2))\le m(v)d(a_1,a_2).$$

For each $a_1,a_2\in A$, $a_1\ge a_2$.

($J_2$) There is a random variable $a_0 : \phi\to A$ with

$$a_0(v)\le T(v,a_0(v)),\ \text{for all v}\in\phi,$$

Or

$$a_0(v)\ge T(v,a_0(v)),\ \text{for all v}\in\phi.$$

Then there is a random variable $a : \phi\to A$ which is a random fixed point of T.

**Corollary 3.6.** Let $(\phi,T)$ be a measurable space and $(A,d,\alpha)$ be a complete partially ordered separable metric space. If $T : \phi\times A\to A$ is a continuous random operator such that, $T(v,\cdot)$ is a monotone (either order-preserving or order-reserving) operator. Suppose that the following conditions hold:

($J_1$) For each $\nu \in \phi$, there exists $m(\nu) \in \left(0, \frac{1}{2}\right)$ such that

$$d(T(\nu; a_1), T(\nu; a_2)) \leq m(\nu)[d(a_1, T(\nu; a_1)) + d(a_2, T(\nu; a_2))].$$

For each $a_1, a_2 \in A,\ a_1 \geq a_2$.

($J_2$) There is a random variable $a_0 : \phi \rightarrow A$ with

$$a_0(\nu) \leq T(\nu, a_0(\nu)),\ \text{for all } \nu \in \phi,$$

Or

$$a_0(\nu) \geq T(\nu, a_0(\nu)),\ \text{for all } \nu \in \phi.$$

Then there is a random variable $a : \phi \rightarrow A$ which is a random fixed point of T.

## 4. Discussion

The theory of random nonlinear integral equations play a significant role in modeling physical phenomena in various branches of mathematics and applied sciences. Many mathematicians have established the existence and uniqueness of a solution for random integral equations via fixed point theorems. For instance, see [3]and [11].

To prove the existence and uniqueness of the solution of random nonlinear integral equation presented as follows

$$a(t;\nu) = h(t;\nu) + \int_S k(t,s;\nu)T(s, a(s;\nu))d\mu(s) \tag{4}$$

We apply Theorem 3.1 and Theorem 3.2 to obtain the result. The following assumptions are made with respect to the random kernel $k(t,s;\nu)$.

Let S be a locally compact metric space with metric defined on $S\times S$ and let $\mu$ be a complete $\sigma$- finite measure defined on Borel subset of S. Suppose S is a countable family of compact subset $\{c_n\}$ having the properties that $c_1 \subset c_2 \subset \cdots$ and for any other compact set S there is a $c_i$ contain in it [10].

**Definition 4.1.** [7]**:** Let $C(S, L_2(\phi, \xi, \mu))$ be the space of all continuous function from S into $L_2(\phi, \xi, \mu)$ with the topology of uniform convergence on compact set S, that is, for each fixed $t \in S$, $a(t;\nu)$ is random variable such that $||a(t;\nu)||^2_{L_2(\phi,\xi,\mu)} = \int_\phi \left|a(t;\nu)\right|^2 d\mu(\nu) < \infty$.

Consider the function $h(t;\nu)$and $T(t, a(t;\nu))$ to be in the space $C(S, L_2(\phi, \xi, \mu))$ concerning the random kernel. We assume that for $(t,s)$, $k(t,s;\nu) \in L_\infty(\phi, \xi, \mu)$ the norm is denoted by

$$\left|\left|\left|k(t,s;\nu)\right|\right|\right| = \left|\left|k(t,s;\nu)\right|\right|_{L_\infty(\phi,\xi,\mu)} = \mu - \operatorname*{ess\,sup}_{\nu\in\phi}\left|k(t,s;\nu)\right|.$$

Suppose we assume $k(t,s;\nu)$ to be $\left|\left|\left|k(t,\ s;\nu)\right|\right|\right|.\left|\left|a(t;\nu)\right|\right|_{L_2(\phi,\xi,\mu)}$ is$\mu$-integrable with respect to S for each $t \in S$ and $a(s;\nu)$ in $C(S, L_2(\phi, \xi, \mu))$ and there is a function J which defined $\mu$ a. e. on S such that $J(s)\left|\left|a(s;\nu)\right|\right|_{L_2(\phi,\xi,\mu)}$ is $\mu$- integrable and for $(t,\ s) \in S \times S$, $\left|\left|\left|k(t,u;\nu) - k(s,u;\nu)\right|\right|\right|.\left|\left|a(t;\nu)\right|\right|_{L_2(\phi,\xi,\mu)} \leq J(u)\left|\left|a(t;\nu)\right|\right|_{L_2(\phi,\xi,\mu)}\mu$ a. e.

Therefore for $(t,s) \in S \times S$, we obtain $k(t,s;\nu) \times (s;\nu) \in L_2(\phi,\xi,\mu)$. We now define the random integral operator

$$[X(\nu)a](t;\nu) = \int_S k(t,s;\nu)a(s;\nu)d\mu(s).$$

Where the integral is a Bochner integral. From the conditions on $k(t,s;\nu)$, we obtain that for each $t \in S$, $[X(\nu)a](t;\nu) \in L_2(\phi,\xi,\mu)$ and $X(\nu)$ is a continuous linear operator from $C(S, L_2(\phi, \xi, \mu))$ into itself.

**Definition 4.2.** [7]**:** Let E and F be Banach spaces. The pair (E, F) is said to be admissible with respect to a random operator, $[X(\nu)a](t;\nu)$ if $[X(\nu)a](t;\nu)(E) \subset F$.

**Lemma 4.3.** [7]**:** If (A) is a continuous linear operator from $C(S, L_2(\phi, \xi, \mu))$ into itself and $E, F \subset C(S, L_2(\phi, \xi, \mu))$ are Banach spaces stronger than $C(S, L_2(\phi, \xi, \mu))$ such that (E, F) is admissible with respect to A, then (A) is continuous from E to F.

The following theorem gives the existence and uniqueness of a random solution of Eq. (4).

**Theorem 4.1.** Let the random integral Eq. (4) be subjected to the following conditions:

(i) E and F are Banach spaces stronger than $C(S, L_2(\phi, \xi, \mu))$ such that (E, F) is admissible with respect to the integral operator defined by (4).

(ii) $a(t;\nu) \rightarrow T(t, a(t;\nu))$ is an operator from the set

$$U(\alpha) = \left\{a(t;\nu) : a(t;\nu) \in F, \left|\left|a(t;\nu)\right|\right|_F \leq \alpha\right\}$$

into the space E satisfying

$$\begin{aligned}&\left|\left|T(t,a(t;\nu)) - T(t,b(t;\nu))\right|\right|_E \leq m(\nu)\left|\left|a(t;\nu) - b(t;\nu)\right|\right|_E \\ &+n(\nu)[\left|\left|a(t;\nu) - T(t,a(t;\nu))\right|\right|_E + \left|\left|b(t;\nu) - T(t,b(t;\nu))\right|\right|_E] \\ &+q(\nu)[\left|\left|a(t;\nu) - T(t,b(t;\nu))\right|\right|_E + \left|\left|b(t;\nu) - T(t,a(t;\nu))\right|\right|_E]a.s.\end{aligned}$$

For $a(t;\nu), b(t;\nu) \in U(\alpha)$, with $m(\nu) + q(\nu) < 1$ , $n(\nu) < 1$ and

(iii) $h(t;\nu) \in F$.

Then (4) has a unique random solution in $U(\alpha)$ provided $k(\nu) < 1$ a.s. and $\left|\left|h(t;\nu)\right|\right|_F + 2k(\nu)\left|\left|T(t;0)\right|\right|_E \leq \alpha(1 - k(\nu))a.s.$where $k(\nu)$ is the norm of $X(\nu)$.

**Proof:** For arbitrary $a_0(t;\nu) \in U(\alpha)$, we choose $a_1(t;\nu) \in U(\alpha)$ such that $a_0(t;\nu) = T(t, a_0(t;\nu)) = a_1(t;\nu)$. This shows that $a_0(t;\nu)$is the solution of the operator Y(v). Assume $a_0(t;\nu) \leq T(t,\ a_0(t;\nu))$we have $a_0(t;\nu) \leq T(t,\ a_0(t;\nu)) = a_1(t;\nu)$. For $a_2(t;\nu) \in U(\alpha)$ we have $a_1(t;\nu) \leq T^2(t,\ a_0(t;\nu)) = a_2(t;\nu)$. Continue the process we have $a_0(t;\nu) \leq T(t, a_0(t;\nu)) = a_1(t;\nu) \leq T^2(t, a_0(t;\nu)) = a_2(t;\nu) \leq \cdots$ - $\leq T^n(t, a_0(t;\nu)) = \cdots$Thus we define a sequence $T^n(t, a_0(t;\nu)) \in U(\alpha)$ such that $T^n(t,\ a_0(t;\nu)) \leq T^{n+1}(t,\ a_0(t;\nu))$. This shows that it is monotone nondecreasing.

Taking $a_0(t;\nu), b_0(t;\nu) \in U(\alpha),\ a_0(t;\nu) < b_0(t;\nu)$we define a metric d on $U(\alpha)$ by $|a_0(t;\nu) - b_0(t;\nu)|$.

This shows that $(U(\alpha), d, \leq)$ is a complete partially ordered separable metric space.

Next we prove that the random nonlinear integral operator is contractive. Suppose the operator $Y(\nu)$ from $U(\alpha)$ into F is defined by

$$[Y(\nu)a](t;\nu) = h(t;\nu) + \int_S k(t,s;\nu)T(s, a(s;\nu))d\mu(s).$$

Then we obtain the following from the conditions of the theorem.

$$\begin{aligned}\left|\left|[Y(\nu)a](t;\nu)\right|\right|_F &\leq \left|\left|h(t;\nu)\right|\right|_F + k(\nu)\left|\left|T(t,a(t;\nu))\right|\right|_D a.s. \\ &\leq \left|\left|h(t;\nu)\right|\right|_E + k(\nu)\left|\left|T(t;0)\right|\right|_E \\ &+ k(\nu)\left|\left|T(t,a(t;\nu) - T(t;0)\right|\right|_E.\end{aligned}$$

But,

$$\begin{aligned}&k(\nu)\left|\left|T(t,a(t;\nu) - T(t;0)\right|\right|_E \leq k(\nu)(m(\nu)\left|\left|a(t;\nu) - 0\right|\right|_E \\ &+n(\nu)[\left|\left|a(t;\nu) - T(t,a(t;\nu))\right|\right|_E + \left|\left|0 - T(t,0)\right|\right|_E] \\ &+q(\nu)[a(t;\nu) - T(t,0)||_E + \left|\left|0 - T(t,a(t;\nu))\right|\right|_E])\end{aligned}$$

$$\le k(v)(m(v)||a(t;v)||_E$$
$$+n(v)[||a(t;v)-T(t,a(t;v))||_E+||T(t,0)||]$$
$$+q(v)[||a(t;v)||_E+||T(t,a(t;v))||_E])$$

$$\le k(v)[(m(v)+q(v))||a(t;v)||_E$$
$$+n(v)||T(t,0)||_E]$$

$$\le k(v)\alpha + k(v)||T(t,0)||_E,$$

Since $m(v)+q(v)<1$ and $n(v)<1$. Hence,

$$||[Y(v)a](t;v)||_F \le ||h(t;v)||_F + k(v)||T(t,0)||_E$$
$$+k(v)\alpha + k(v)||T(t,0)||_E$$
$$\le ||h(t;v)||_E + 2k(v)||T(t,0)||_E + k(v)\alpha a.s.$$
$$\le \alpha(1-k(v)) + k(v)\alpha \text{ a.s.}$$
$$< \alpha.$$

Therefore, $[Y(v)a](t;v) \in U(\alpha)$.

If $a(t;v), b(t;v) \in U(\alpha)$ then using condition (ii) we obtain,

$$||[Y(v)a](t;v) - [Y(v)b](t;v)||_F$$
$$= \left|\left| \int_S k(t,s;v)[T(s,a(s;v)-T(s,b(s;v)\right] d\mu(s)\right| |_E$$
$$\le k(v)||T(t,a(t;v)-T(t,b(t;v))||_E \text{ a.s..}$$

$$\le k(v)(m(v)||a(t;v)-b(t;v)||_E$$
$$+n(v)[||a(t;v)-T(t,a(t;v))||_E$$
$$+||b(t;v)-T(t,b(t;v))||_E]$$
$$+q(v)[a(t;v)-T(t,b(t;v))||_E$$
$$+||b(t;v)-T(t,a(t;v))||_E]) \text{ a.s.}$$

$$\le m(v)||a(t;v)-b(t;v)||_E$$
$$+n(v)[||a(t;v)-T(t,a(t;v))||_E$$
$$+||b(t;v)-T(t,b(t;v))||_E]$$
$$+q(v)[a(t;v)-T(t,b(t;v))||_E$$
$$+||b(t;v)-T(t,a(t;v))||_E] \text{ a.s.}$$

Since $k(v)<1$ a.s. Therefore Y(v) is a random nonlinear contraction operator on $U(\alpha)$. By Theorem 3.1 and 3.2, there is a random variable $y: \phi \to B$ which is the unique random solution of the operator (4).

## 5. Conclusion

This research proved the existence and uniqueness of random fixed point for certain contractive mappings in complete partially ordered separable metric spaces. The existence and uniqueness of a random solution for nonlinear integral equation is established using this contractive operator.

## Declarations

### Author contribution statement

K.S. Eke; Wrote the paper.
H. Akewe; Conceived and designed the experiment.
S.A. Bishop; Analyzed and interpreted the data.

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### Additional information

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